Geodesics on Surfaces of Revolution

General Theory Applied to Paraboloid & Hexenhut

Nicholas Wheeler
February 2016

Introduction. The following material, borrowed from recent writings, is presented as it relates specifically to description of geodesics on two closely related surfaces, the unit paraboloid and the unit hexenhut.

Surfaces of revolution. For surfaces of revolution Σ we have in general the parameterization

$$\boldsymbol{r} = \begin{pmatrix} q(u)\cos v \\ q(u)\sin v \\ p(u) \end{pmatrix} \tag{1.1}$$

where $q(u)$ is the cross-sectional radius of Σ at $z = p(u)$, but in many cases find it more convenient to work from the simpler parameterization

$$\boldsymbol{r} = \begin{pmatrix} r(u)\cos v \\ r(u)\sin v \\ u \end{pmatrix} \tag{1.2}$$

The 1^{st} and 2^{nd} fundamental forms supply

$$\mathbb{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_u\boldsymbol{\cdot}\boldsymbol{r}_v \\ \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_u & \boldsymbol{r}_v\boldsymbol{\cdot}\boldsymbol{r}_v \end{pmatrix} = \begin{pmatrix} p_u^2 + q_u^2 & 0 \\ 0 & q^2 \end{pmatrix} \tag{2.1}$$

$$\begin{aligned}\mathbb{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{r}_{uu}\boldsymbol{\cdot}\boldsymbol{N} & \boldsymbol{r}_{uv}\boldsymbol{\cdot}\boldsymbol{N} \\ \boldsymbol{r}_{vu}\boldsymbol{\cdot}\boldsymbol{N} & \boldsymbol{r}_{vv}\boldsymbol{\cdot}\boldsymbol{N} \end{pmatrix} \\ &= \frac{1}{\sqrt{p_u^2 + q_u^2}} \begin{pmatrix} (q_u p_{uu} - p_u q_{uu}) & 0 \\ 0 & q\,p_u \end{pmatrix}\end{aligned} \tag{2.2}$$

where $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v / |\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is the unit normal at $\boldsymbol{r}(u, v)$. The differential geometry of surfaces of revolution owes much of its distinctive simplicity to the circumstance that the matrices $\mathbb{G}$ and $\mathbb{H}$ are *diagonal* and *v-independent*.

The v-independent Gaussian curvature on such surfaces Σ is given by

$$K = \frac{\det \mathbb{H}}{\det \mathbb{G}} = \frac{p_u(q_u p_{uu} - p_u q_{uu})}{q(p_u^2 + q_u^2)^2} \tag{3.1}$$

which in the simplified parameterization (1.2) becomes

$$K = -\frac{r_{uu}}{r(1 + r_u^2)^2} \tag{3.2}$$

The geodesic differential equation. Let $x^i(t)$ describe a t-parameterized curve

$$\mathcal{C} : x^i(t_1) \longrightarrow x^i(t_2)$$

in a metrically connected n-dimensional manifold: $i = 1, 2, \ldots, n$ The length of such a curve is given by

$$S[x(t)] = \int_{t_1}^{t_2} \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}\, dt$$

Geodesics (which were reportedly given their name by Liouville) are curves which are in the variational sense extremal:[1]

$$\delta S[x(t)] \equiv S[x(t) + \delta x(t)] - S[x(t)] = 0$$

For curves inscribed on $\{u, v\}$-parameterized surfaces Σ (which is to say: on 2-dimensional manifolds that inherit their metric structure from the Euclidean structure of the enveloping 3-space) we have (with $x^1 = u, x^2 = v$)

$$S[x(t)] = \int_{t_1}^{t_2} \sqrt{g_{11}\dot{u}\dot{u} + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}\dot{v}}\, dt$$

When Σ is a surface of revolution it was seen at (2.1) that the g_{12} term drops away, and if (instead of generic t-parameterization) we adopt the natural u-parameterization of (1) we have still more simply

$$S[x(t)] = \int_{t_1}^{t_2} \sqrt{g_{11}(u) + g_{22}(u)v_u v_u}\, du$$

The Euler-Lagrange equation now reads

$$\left\{\frac{d}{du}\frac{\partial}{\partial v_u} - \frac{\partial}{\partial v}\right\}\sqrt{g_{11}(u) + g_{22}(u)v_u v_u} = 0 \tag{4.1}$$

which, because $\sqrt{\text{etc.}}$ is v-independent, supplies the immediate first integral

$$\frac{g_{22}v_u}{\sqrt{g_{11} + g_{22}v_u v_u}} = c \tag{4.2}$$

where the constant c is a geometric analog of the conserved angular momentum of axially symmetric mechanical systems.

[1] Tullio Levi-Civita (1917) recognized that *parallel transport of a tangent* serves alternatively to produce geodesic curves, provided the affine connection used to define covariant differentiation (whence also intrinsic differentiation and parallel transportation) is the Christoffel connection $\Gamma^i{}_{jk}$, which is an object constructed from the metric tensor g_{ij} and its first partial derivatives.

From (4.2) it follows that

$$\frac{dv(u)}{du} = \pm c \frac{\sqrt{g_{11}}}{\sqrt{g_{22}(g_{22} - c^2)}} \equiv \pm F(u; c) \tag{5}$$

and therefore that

$$v(u) = \pm \int^u F(\hat{u}; c)\, d\hat{u} \tag{6}$$

As Luther Eisenhart remarks,[2] "the geodesics upon a surface of revolution referred to its meridians and parallels can be found by quadrature."[3] There is, however, no guarantee that the integral (6) is tractable = describable in terms of named functions, and in the case of the hexenhut we will find that it is not.

Clairaut's Theorem. Write

$$\boldsymbol{r}(u) = \begin{pmatrix} r(u)\cos v(u) \\ r(u)\sin v(u) \\ u \end{pmatrix}$$

to describe a u-parameterized curve $\mathcal{C}$ inscribed on Σ. The vector

$$\boldsymbol{t} = \tfrac{d}{du}\boldsymbol{r}(u) = \boldsymbol{r}_u + v_u \boldsymbol{r}_v$$

is tangent to $\mathcal{C}$ at u. It's squared length is $\boldsymbol{t}\cdot\boldsymbol{t} = g_{11} + g_{22}v_u^2$ so the unit tangent is

$$\boldsymbol{T} = \frac{\boldsymbol{r}_u + v_u \boldsymbol{r}_v}{\sqrt{g_{11} + g_{22}v_u^2}}$$

The vector

$$\boldsymbol{r}_v = \begin{pmatrix} -r\sin v \\ r\cos v \\ 0 \end{pmatrix}$$

is tangent to the $\mathcal{R}_u$, the encircling parallel at u, and when normalized becomes

$$\boldsymbol{S} = r^{-1}\boldsymbol{r}_v$$

From these remarks it follows that

$$\boldsymbol{S}\cdot\boldsymbol{T} = \cos\alpha = \frac{\boldsymbol{r}_v\cdot(\boldsymbol{r}_u + v_u\boldsymbol{r}_v)}{r\sqrt{g_{11} + g_{22}v_u^2}} = \frac{g_{22}v_u}{r\sqrt{g_{11} + g_{22}v_u^2}} \tag{7}$$

where α is the angle subtended by $\boldsymbol{S}$ and $\boldsymbol{T}$; *i.e.*, the angle evident at the point where the curves $\mathcal{C}$ and $\mathcal{R}_u$ intersect. But if the curve $\mathcal{C}$ in question is geodesic

---

[2] *A Treatise on the Differential Geometry of Curves and Surfaces* (1909), page 206.

[3] By "parallels" Eisenhart means "cross-sectional circles of constant u." "Meridians"—curves of constant v—are geodesics that arise from (5) in the case $c = 0$.

we have (4.2), which in conjunction with (7) supplies

$$r\cos\alpha = c \tag{8}$$

Which is **Clairaut's Theorem**, first remarked by Alexis Claude Clairaut (1713–1765) in a work (*Théorie de la figure de la terre, tirée des principes de l'hydrostatique*, 1743) devoted mainly to other things.[4] I do not know the line of argument that led Clairaut to his celebrated theorem, but it must certainly have differed from the one employed above.[5]

From Clairaut's Theorem (8) it follows that as $r(u)$ increases the c-geodesic veers toward (runs more nearly parallel to) the meridians, and as $r(u)$ decreases veers away from the meridians (α decreases, $\cos\alpha$ increases). Geodesics can intersect, can in particular cross meridians. But they can never become tangent to one another (or to a meridian), for the solutions of (5) cannot bifurcate.

Siblings: the paraboloid and the hexenhut. The unit paraboloid $z = x^2 + y^2$ arises from (1.2) from setting $r(u) = \sqrt{u}$, the unit hexenhut $z^{-1} = x^2 + y^2$ from setting $r(u) = 1/\sqrt{u}$, and it is in this obvious sense that they can be said to be "siblings." On the paraboloid $r(u)$ ranges $0 \to \infty$, while on the hexenhut $r(u)$ ranges $\infty \to 0$ as one ascends up the surface; *i.e.*, as $u = z$ ranges $0 \to \infty$. Clairaut's Theorem asserts that on the paraboloid every c-geodesic ($c \neq 0$) veers *toward* the meridians ($\alpha \to \frac{1}{2}\pi$), while on the hexenhut every such geodesic veers *away* from the meridians ($\alpha \to 0$), as $u \to \infty$. In the

[4] Clairaut, who had accompanied Maupertuis to Lapland on the French Meridian Expedition of 1736, was concerned in that work with marshaling evidence that the earth is an oblate spheroid. But it is not surprising that the work included remarks relating to the theory of curves: he had written on the subject already at the age of twelve, and the next year read a paper on the subject before the Académie Française. The publication of a paper on the theory of certain "tortuous curves" in 1731 led to his election to the French Academy of Sciences at the age of eighteen. Clairaut made significant contributions also celestial mechanics (three-body problem, theory of the moon, predicted return in 1759 of Halley's comet) and to pure and applied mathematics (discrete Fourier transform). But Charles Bossut—a man of similar interests, but separated in age as was Richard Crandall from me—wrote that "He was focused with dining and with evenings, coupled with a lively taste for women, and seeking to carry his pleasures into his day-to-day work he lost rest, health, and finally life at the age of fifty-two."

[5] Problems involving what we now call the "calculus of variations" were considered already by Newton and Jacob Bernoulli, but this class of problems was first addressed in a general systematic way by Euler in 1733, and the work of Euler-Lagrange appeared only in the 1750s—too late to have been of any assistance to Clairaut. It is interesting that Lagrange's contributions to the calculus of variations drew inspiration partly from the physical intuition of Maupertuis, an idea—anticipated already by Leonardo da Vinci—to which he gave the name "Principle of Least Action."

paragraphs that follow I look comparatively to the geodesic details in those two cases.

GEODESICS ON THE UNIT PARABOLOID

Introducing $p(u) = u, q(u) = r(u) = \sqrt{r}$ into (2.1) we obtain

$$\mathbb{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(1+4u)/u & 0 \\ 0 & u \end{pmatrix} \tag{9.1}$$

which by $\det \mathbb{G} = \frac{1}{4}(1+4u)$ is non-singular for all u. The curvature, by (3.2), is given by

$$K = \frac{4}{(1+4u)^2} \tag{9.2}$$

which is positive for all u but vanishes in the limit $u \to \infty$: the paraboloid is asymptotically flat. The geodesic equation (5) reads

$$v_u = \pm\tfrac{1}{2}c\sqrt{\frac{1+4u}{u^2(u-c^2)}} \quad : \quad \text{real if and only if } u \geqslant c^2 \tag{9.3}$$

Noting that $u_v = (v_u)^{-1}$ vanishes at $u = c^2$ (which is to say: Clairaut's angle $\alpha = 0$ at $u = c^2$ where $r(u) = c$, precisely as asserted by Clairaut's Theorem) we conclude that c-geodesics on the the paraboloid exhibit a "turning point" at $u = c^2$, where they become tangent to a parallel and below which they do not descend.

Robert Weinstock asserts,[6] and *Mathematica* confirms, that the solution of (9.3) can be described

$$v(u) = \pm\Big\{\text{arcsin}\Big[\frac{u-c^2}{u(1+4c^2)}\Big]^{\frac{1}{2}} + 2c\log\Big[k\Big(2\sqrt{u-c^2}+\sqrt{4u+1}\Big)\Big]\Big\} \tag{9.4}$$

where k is a constant of integration. For $u \gg c^2$ we therefore have

$$v(u) \approx \pm\Big\{C + c\log u\Big\} \quad \text{with} \quad C = \text{arcsin}\Big[\frac{1}{(1+4c^2)}\Big]^{\frac{1}{2}} + 2c\log 4k \tag{9.5}$$

This function grows slowly but without bound, meaning that every c-geodesic ($c \neq 0$) wraps around the paraboloid (the $\pm$ determines the sense: $\circlearrowleft$ or $\circlearrowright$) and crosses every meridian infinitely many times. It was an alternative proof of this fact—made somewhat surprising by the circumstance that every ascending geodesic becomes (by Clairaut's Theorem) ever more nearly tangent to the meridians—that engaged Ahmed Sebbar's attention and inspired this entire effort.[7]

---

[6] *Calculus of Variations, with Applications to Physics & Engineering* (1952, Dover 1974), Exercise 5, page 45.

[7] Manfredo P. do Carmo, *Differential Geometry of Curves and Surfaces* (1976), Example 6, pages 258–260. This splendid text is available on the web as a free download.

The following functions

$$v(u) = v_0 \pm \left\{ \arcsin\left[\frac{u-c^2}{u(1+4c^2)}\right]^{\frac{1}{2}} + 2c\log\left[\frac{2\sqrt{u-c^2}+\sqrt{4u+1}}{\sqrt{1+4c^2}}\right]\right\}$$

—which differ from (9.4) only by additive constants—possess the property that they join smoothly at the turning point $\{u = c^2, v = v_0\}$. They can be considered to describe a *single* geodesic that winds down the paraboloid, turns at the turning point and then winds back up again, crossing itself infinitely many times. Continuation through the turning point becomes quite natural if one adopts the parallel transport approach to the theory of geodesics. With this development it becomes possible in principle to describe the geodesic that links specified endpoints $\{u_1, v_1\} \rightarrow \{u_2, v_2\}$: this amounts to assigning specific values to v_0 and c. Since $\{u, v + n2\pi\}$ $(n = 0, \pm1, \pm2, \ldots)$ all refer to the same point on the paraboloid, we conclude that in fact *infinitely many* distinct geodesics link any pair of specified endpoints. In a more protracted discussion one could look to the *locus of the turning points* of the geodesics that pass through a specified point in all possible directions.

GEODESICS ON THE UNIT HEXENHUT

Introducing $p(u) = u, q(u) = r(u) = 1/\sqrt{r}$ into (2.1) we obtain

$$\mathbb{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} (1+4u^3)/4u^3 & 0 \\ 0 & 1/u \end{pmatrix} \tag{10.1}$$

which by $\det \mathbb{G} = (1+4u^3)/4u^4$ is seen to be singular at $u = 0$ (which we agree to avoid) and to vanish as $u \rightarrow \infty$: asymptotically the hexenhut hugs the z-axis ever more closely, and (in effect) ultimately "loses a dimension." The Gaussian curvature, by (3.2), becomes

$$K = -\frac{12u^4}{(1+4u^3)^2} \tag{10.2}$$

which vanishes at $u = 0$ and also as $u \rightarrow \infty$; it assumes its extreme value

$$K_{\text{extreme}} = -\tfrac{1}{3}2^{\frac{2}{3}} = -0.5291 \quad \text{at} \quad u = (1/2)^{\frac{1}{3}} = 0.7937$$

The geodesic differential equation (5) has become

$$v_u = \pm\tfrac{1}{2}c\sqrt{\frac{1+4u^3}{u(1-c^2u)}} \quad : \quad \text{real if and only if } u \leqslant c^{-2} \tag{10.3}$$

Noting that $u_v = (v_u)^{-1}$ vanishes at $u = c^{-2}$ (which is to say: Clairaut's angle $\alpha = 0$ at $u = c^{-2}$ where $r(u) = c$, precisely as asserted by Clairaut's Theorem) we might conclude that c-geodesics on the the hexenhut exhibit a "turning point" at $u = c^{-2}$, where they become tangent to a parallel and above which they do not rise. It will emerge, however, that the phrase "turning point" somewhat misrepresents the situation, for the turning that goes on there is asymptotically very leisurely; it might better be called "hesitation point," a "point of asymptotic statis." But it is in any event clear that a c-geodesic will ascend to high altitudes only if c is small.

From (10.3) we obtain this instance of (6):

$$v(u) = \pm\int^u F(\hat{u};c)\,d\hat{u} \quad : \quad F(u;c) = \tfrac{1}{2}c\left(\frac{1+4u^3}{u(1-c^2u)}\right)^{\frac{1}{2}} \tag{10.4}$$

But the integral is not listed in Gradshteyn & Ryzhik[8], elicits no response from *Mathematica*. Graphic experimentation leads, however, to the observation that

$$\begin{aligned} F(u;c) &= f(u;c)\cdot\sqrt{1+\frac{1}{4u^3}} \ : \ f(u;c) \equiv \tfrac{1}{2}c\left(\frac{4u^3}{u(1-c^2u)}\right)^{\frac{1}{2}} = cu\frac{1}{\sqrt{1-c^2u}} \\ &= f(u;c)\cdot\left\{1+\frac{1}{8u^3}-\frac{1}{128u^6}+\frac{1}{1024u^9}-\frac{5}{32768u^{12}}+\cdots\right\} \end{aligned}$$

and that for $u > (\frac{1}{8})^{\frac{1}{3}} = \frac{1}{2}$ the function $f(u;c)$ provides an excellent and rapidly ever better approximation to $F(u;c)$. And its integral is not only tractable, it is elementary:

$$v(u) = \pm\int^u f(\hat{u};c)\,d\hat{u} = v_0 \mp \frac{2(2+c^2u)\sqrt{1-c^2u}-4}{3c^3} \tag{10.5}$$

From

$$|v_u(u)| = f(u;c) = cu + \tfrac{1}{2c}(c^2u)^2 + \tfrac{3}{8c}(c^2u)^3 + \tfrac{5}{16c}(c^2u)^4 + \cdots$$

we see that v_u grows linearly until the condition $u \ll c^{-2}$ is violated, then grows ever more rapidly than linearly, until as $u \to c^{-2}$ its rate of growth $\to \infty$. The ultimate angular advance of the c-geodesic is (in the approximation (10.5))

$$\text{total angular advance} = v(c^{-2}) = \tfrac{4}{3}c^{-3} \tag{10.6}$$

by which point it has completed a finite

$$\text{number of circuits around the hexenhut} = \tfrac{4}{3}c^{-3}/2\pi \tag{10.7}$$

The "pitch" $\wp(u;c)$ of the twisting geodesic (u-advance per circuit) is at high altitudes well approximated by

$$\wp(u;c) = 2\pi\,u_v = 2\pi\,(v_u)^{-1} = 2\pi\frac{\sqrt{1-c^2u}}{cu}$$

Evidently $\wp(u;c) \to 0$ as $u \to$ turning point at c^{-2}; this is consistent with Clairaut's Theorem, which requires that geodesics on surfaces of revolution be tangent to the parallel at their turning points. From

$$\begin{aligned} u_{vv} = (v_u)^{-1}_u\cdot u_v = (v_u)^{-1}_u\cdot(v_u)^{-1} &= c^4\frac{1}{(c^2u)^2}\left(\frac{1}{2}-\frac{1}{c^2u}\right) \\ &= -\tfrac{1}{2}c^4 \text{ at the turning point} \end{aligned}$$

we learn that the c-geodesic is prepared to execute a downward turn (but a

[8] *Table of Integrals, Series, and Products* (1965).

very leisurely one when c^{-2} is large; *i.e.*, at high altitudes, where the radius c of the hexenhut is small) when it finally achieves its turning point

At (10.5) we encounter two distinct families of v_0-parameterized c-geodesics, the respective members of which will be denoted $v_\pm(u\,;c,v_0)$:

$$\left.\begin{aligned} v_+(u\,;c,v_0) &= v_0 + \frac{2(2+c^2u)\sqrt{1-c^2u}-4}{3c^3} \\ v_-(u\,;c,v_0) &= v_0 - \frac{2(2+c^2u)\sqrt{1-c^2u}-4}{3c^3} \end{aligned}\right\} \tag{10.8}$$

where $v_+(u\,;c,v_0)$ winds $\circlearrowleft$, $v_-(u\,;c,v_0)$ winds $\circlearrowright$. Let it be required that

$$v_+(u\,;c,v_0) = v_-(u\,;c,v_0+\vartheta) \quad \text{at their mutual turning point}$$

Immediately $\vartheta = -\frac{8}{3}c^{-3}$. We conclude that $v_+(u\,;c,v_0)$ and $v_-(u\,;c,v_0 - \frac{8}{3}c^{-3})$ describe smoothly-joined ascending/descending branches of the same "complete geodesic."

From this argument—if we accept the inoffensive approximation from which it proceeds— it follows that geodesics on the hexenhut intersect meridians (and themselves) only a finite number

$$\text{number of meridian crossings} = \lfloor \tfrac{8}{3}c^{-3}/2\pi \rfloor$$

of times, and that Sebbar's conjecture (namely: that on the hexenhut, as on the paraboloid, the number of meridian crossings is infinite) to be untenable.

To describe the $\circlearrowleft$ geodesic that links specified endpoints $\{u_1,v_1\}\rightarrow\{u_2,v_2\}$ one must solve the simultaneous equations

$$\begin{aligned} v_1 &= v_0 + \frac{2(2+c^2u_1)\sqrt{1-c^2u_1}-4}{3c^3} \\ v_2 &= v_0 + \frac{2(2+c^2u_2)\sqrt{1-c^2u_2}-4}{3c^3} \end{aligned}$$

for v_0 and c. The solution provided by *Mathematica* is enormously complicated. But for small c we in lowest order have simply

$$\begin{aligned} v_1 &= v_0 - \tfrac{1}{2}cu_1^2 \\ v_2 &= v_0 - \tfrac{1}{2}cu_2^2 \end{aligned}$$

which give

$$v_0 = \frac{u_2^2v_1 - u_1^2v_2^2}{u_2^2 - u_1^2} \quad \text{and} \quad c = 2\,\frac{v_1 - v_2}{u_2^2 - u_1^2} \tag{10.9}$$

From the fact that the equations (10.9) are not invariant under $v_2 \rightarrow v_2 + 2\pi n$ we conclude that there are *indefinitely many* geodesics that link $\{u_1,v_1\}\rightarrow\{u_2,v_2\}$, and a similar population of $\circlearrowright$ geodesics. The multiplicity of such geodesics—distinguished from one another by their $\pm$ "winding numbers"—would appear to be (as for the cylindar and pseudosphere) implicit in the topology of the hexenhut. I speculate that no such geodesic can possess a turning point, unless it terminates at one.

A word about the figures. The figures were constructed by *Mathematica* v7.0.0, and are best viewed in the notebook (which I ship together with this pdf file), for there they can be viewed from all angles and manipulated at will. I have refrained from attaching pdf versions of the figures to the text because, as I have discovered, illustrations produced by such old software remain invisible on some more recent platforms. But I have included a file containing both pdf and jpeg renditions of the figures in the hope that one or the other will serve its intended purpose.

CAPTIONS

1. Geodesic on Unit Paraboloid. The altitude parameter ranges $0 \leqslant u \leqslant 10$. Inscribed on the paraboloid is the "complete geodesic" created by the smooth joining of $\circlearrowright$ and $\circlearrowleft$ c-geodesics with $c = 1$.

2. Geodesic Crossings on Unit Paraboloid. The range of the preceding figure has been extended to $0 \leqslant u \leqslant 120$, far enough to show the 2^{nd} self-intersection of the geodesic (of whch there will ultimately be infinitely many). The view is fromhigh on the z-axis. One is not surprised to see that successive crossings occur on opposite meridians.

3. Geodesic Near Base of Hexenhut. The altitude parameter ranges $0 \leqslant u \leqslant 10$. Inscribed on the hexenhut is a $\circlearrowleft$ c-geodesic with $c = 0.1$. The geodesic derives from (10.8), which is accurate only for $u \gg 1$.

4. Hexenhut Geodesic at Turning Point. The geodesic shown above has a turning point at $u = c^{-2} = 100$. In the figure $90 \leqslant u \leqslant 110$ and the $\{x, y\}$ coordinates have been rescaled by a factor of 20. The geodesic encircles the hexenhut a total of $\frac{8}{3}c^{-3}/2\pi = 424.413$ times.